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Estimating Temperature-Dependent Thermal Properties

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Parameter estimation techniques are applied to estimate temperature-dependent thermal properties from a series of transient experiments. Several experiments with one- and two-dimensional heat flow that cover a range from room temperature to 500°C are analyzed. Temperature-dependent thermal properties are estimated by connecting the independent experiments in the series during the analysis. The techniques are applied to estimate effective properties of carbon-carbon composite. The temperature dependence of two components of thermal conductivity and volumetric heat capacity are estimated to characterize the assumed orthotropic material. The techniques can be equally applied to other homogenous materials. Combining experiments during the analysis is referred to as a sequential analysis, which uses the concepts of regularization and prior information. Regularization controls variations in the estimated parameters. Prior information carries information from a previous analysis into a subsequent analysis. The functional dependence of the properties estimated with a sequential analysis show excellent agreement with previous analyses that considered the experiments independently.

Nomenclature

$\boldsymbol{b}_n$, $b_{j,n}$	= parameter vector for experiment n
$\hat{\boldsymbol{b}}_n$, $\hat{b}_{j,n}$	= estimated parameter vector, experiment n
C	= specific heat, J/kg C
$\boldsymbol{H}$	= regularization matrix; Eq. (14)
J	= number of sensors
k	= thermal conductivity, W/m°C
N	= number of discrete measurements in time
p	= number of parameters
S	= sum-of-squares function
S_{p_n}	= regularization sum-of-squares function, experiment n
T	= temperature, °C
T_b	= scaled sensitivity coefficient, °C
T_i	= initial temperature, °C
T_j	= discrete temperature value, °C
$\hat{\boldsymbol{T}}_n$, $\hat{T}_{i,j,n}$	= calculated temperature, experiment n, °C
$\boldsymbol{U}_n$, $U_{i,j,n}$	= weighting for prior information, experiment n
$\boldsymbol{u}_n$, $u_{j,n}$	= prior information, experiment n
$\boldsymbol{W}_n$, $W_{i,j,n}$	= weighting for sensors, experiment n, °C^{-2}
$\boldsymbol{X}_n$, $X_{i,j,n}$	= sensitivity coefficient, experiment n
$\boldsymbol{Y}_n$, $Y_{i,j,n}$	= measured temperature, experiment n, °C
α	= regularization weighting
θ	= linear basis function; Eq. (11)
ρ	= density, kg/m^3

Subscripts

cc	= carbon-carbon composite material
x	= x direction
y	= y direction

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I. Introduction

Motivation

ACCURATE modeling of thermal systems is becoming increasingly important. Designers are relying more on computer simulations to design complex thermal systems, with less dependence on costly experimental testing and validation. Consequently, material properties and estimation techniques are required to support simulation-based designs. Methods to estimate thermal properties should be general enough to treat multidimensional cases, anisotropic effects, and temperature dependence. In this paper procedures are described for estimating temperature-dependent thermal properties (thermal conductivity and volumetric heat capacity). A series of discrete independent experiments that cover a desired temperature range are analyzed in a sequential manner to estimate a functional dependence for the thermal properties. Issues associated with a sequential analysis and the benefit of such an analysis are discussed.

Several experimental approaches are possible for estimating temperature-dependent thermal properties. One approach involves conducting experiments at regular intervals over the temperature range. Each experiment is conducted such that the change in the thermal properties with temperature for the experiment is negligible; the thermal properties are assumed to be constant for that experiment. These experiments are analyzed independently, and estimated thermal properties are obtained for each experiment. Temperature dependence of the thermal properties is obtained through a post-processing (curve fit) of the thermal properties estimated from the individual experiments. In this approach the temperature dependence of the thermal properties depends on the reference temperature selected for the experiment. Initial temperature and average temperature (in space, with time, or both) are possible selections for a reference temperature. Previous experimental studies to estimate temperature-dependent properties have applied this type of approach.[1–3]

A second approach involves conducting a single experiment with a temperature range that spans the entire range desired for the thermal properties. An advantage of this approach is that physical changes in the material can be modeled with the temperature dependence of the properties.[4] The single experiment is analyzed to directly estimate the dependence of the thermal properties. Practical problems exist for conducting experiments of this type when a

large temperature range is desired. For example, it can be difficult to cover the entire temperature range in a single experiment. Furthermore, heat losses to the ambient environment, which are difficult to quantify, may become significant in an experiment with a large temperature range. Finally, ancillary materials in the experimental apparatus may also become important in the model, and hence must be accurately described.

A third approach lies between the previous two approaches. Individual experiments are conducted at intervals across the desired temperature range, but the properties may vary with temperature for a single experiment. Unless the analysis links the individual experiments, discrete functions from each experiment are obtained. It is not necessarily true that these discrete functions are continuous or will agree for each experiment. It may be difficult to make use of the independent results in this situation.

In a laboratory setting there is sufficient flexibility for performing all such experiments. In the first and second approaches the temperature dependence of the thermal properties is obtained over the entire range without a need to link the analysis of individual experiments. The approaches require covering either many small temperature intervals or a single large temperature interval. In both cases there are already noted disadvantages. Because experiments are costly and time intensive, covering larger intervals would be advantageous. However, experimental accuracy may limit the temperature range of an experiment and require more moderate temperature ranges. Furthermore, if experiments are performed over a long period (years, for instance), it would be beneficial to combine previous experiments with the current without repeating the analysis of all past experiments.

It is therefore desirable to connect a series of experiments during the analysis to estimate temperature-dependent thermal properties. Such an analysis allows discrete experiments, each providing information concerning a portion of the overall range, to be joined in the analysis. This procedure is called a sequential analysis. Although it is unclear whether it is better to conduct one experiment that covers the entire range or a series of experiments, the latter is experimentally simpler. In instances where it is not possible to cover the entire temperature range, this procedure provides a means to link experiments.

An alternative to a sequential analysis is to analyze all experiments simultaneously. This approach would ideally be the most accurate. There are several difficulties with simultaneously analyzing the experiments, as well as benefits not realized in a simultaneous analysis. These aspects are discussed in Sec. II.

The sequential analysis concept, and incorporating information from prior experiments into succeeding ones, applies to the more general case. As more ambitious experimental investigations are conducted, it is becoming increasingly difficult to estimate all desired parameters from a single experiment. In complex cases a series of experiments may be needed to fully characterize all parameters in the model. In such a series of experiments, each individual experiment focuses on one or more parameters. The techniques presented in this paper are applicable for combining or linking the individual experiments during the analysis to characterize the overall model, which is a powerful procedure that allows complex experiments to be combined with complex analyses.

There are two objectives of this paper. The first objective is to outline and discuss the procedure for a sequential analysis. The methods are motivated by a need for connecting a series of experiments to characterize thermal properties. It is, however, not just limited to determining temperature-dependent thermal properties. The second objective is the application of these methods to estimate temperature-dependent thermal properties for a carbon-carbon composite over a temperature range from room temperature to 500°C. Both one- and two-dimensional experiments are analyzed to estimate thermal properties. Because the data has been previously analyzed by considering the experiments independently, we show that consistent results are obtained with a sequential analysis.

Although a sequential analysis has previously been performed,[5] in this paper more experiments (over 20) are considered. In addition, experiments with one- and two-dimensional heat flow are analyzed in a combined manner. This is the first known analysis that combines experiments with one- and two-dimensional heat flow.

In the next section a brief review of the literature is given. Methods used for the sequential analysis are discussed in Sec. II. Aspects of the experimental apparatus and representative experimental data are described in Sec. III. The estimated temperature-dependent properties are presented in Sec. IV, with a comparison to previous analyses. Concluding remark are given in Sec. V.

Literature Review

Previous inverse approaches to estimate temperature-dependent thermal properties have mainly used simulated data. Although these studies are insightful, the data that is assumed available may be excessive. In such investigations measurements are assumed to be available at equally spaced locations (in time and space) throughout the entire domain.[6–8] Such measurements are not generally available from an experiment. Measurements at a relatively small number of locations[4] on an accessible surface[9] typically are available.

An alternate approach to estimate temperature-dependent thermal properties is to assume the properties are a distributed parameter varying in space and time.[7,8] The heat-conduction problem is linear with this assumption. However, the estimation problem is more ill-conditioned because thermal properties are estimated as a function of time and space. For practical problems possibly thousands of components are needed to represent the distributed parameter. This requires measurements throughout the spatial domain as well. Additionally, assuming a distributed parameter does not necessarily insure a one-to-one correspondence between the thermal property and temperature. The temperature dependence of the thermal property could be multivalued. Extending this approach to multidimensional problems, as considered here, is probably impractical because of the data requirements.

II. Analysis

Parameter estimation techniques are powerful analysis methods. In a sequential analysis an adaptation of the techniques and insight to the problem are used. By adapting parameter estimation and combining with regularization methods, analyzing a series of discrete experiments in a sequential manner and carrying information from previous analyses into a succeeding analysis is possible. A typical application of parameter estimation minimizes a sum-of-squares function using data from a single experiment to estimate parameters. For example, the following sum of squares is minimized to estimate parameters from a single experiment:

$$S = (\boldsymbol{Y} - \hat{\boldsymbol{T}})^T \boldsymbol{W} (\boldsymbol{Y} - \hat{\boldsymbol{T}}) = \sum_{j=1}^{J} \sum_{i=1}^{N} W_{i,j} (Y_{i,j} - \hat{T}_{i,j})^2 \qquad (1)$$

where $\boldsymbol{Y}$ and $\hat{\boldsymbol{T}}$ are vectors of measured and calculated temperature composed of measurements at different times and sensor locations, $Y_{i,j} \equiv Y(t_i, \boldsymbol{x}_j)$ and $\hat{T}_{i,j} \equiv \hat{T}(t_i, \boldsymbol{x}_j)$. The weighting matrix is $\boldsymbol{W}$ with components $W_{i,j}$ (often the identity matrix for lack of additional statistical information).

An alternate approach to a sequential analysis is to simultaneously analyze the set of experiments. In this approach an additional summation over the number of experiments is added to Eq. (1). The difficulty is to simultaneously analyze many experiments (over 20 experiments are considered in this paper). It could be cumbersome to analyze more than a handful of experiments simultaneously. There are tools[10] that may aid in such a process, however.

The most beneficial aspect of sequentially analyzing experiments, instead of a simultaneous analysis, is the insight provided while experiments are ongoing. Experiments are generally conducted in a serial manner. A sequential analysis allows an individual experiment to be combined with previous experiments, which provides feedback during the analysis. The feedback can show the impact of a current experiment and indicate the path that future experiments should take.

In addition, recently performed experiments can be combined with past experiments, conducted perhaps last year, without repeating the analysis for all experiments.

To sequentially analyze a series of experiments, without having to reduce the data from all experiments simultaneously, another sum-of-squares function is used[5]:

$$S_n = (\boldsymbol{Y}_n - \hat{\boldsymbol{T}}_n)^T \boldsymbol{W}_n (\boldsymbol{Y}_n - \hat{\boldsymbol{T}}_n) + S_{p_n} \tag{2}$$

The subscript n indicates the sum-of-squares function is for the analysis of the nth experiment. The first term is the same as the preceding function using data from experiment n. An additional prior information/regularization term is added to the sum-of-squares function, for $n = 1$,

$$S_{p_1} = (\boldsymbol{H}\boldsymbol{b}_1)^T (\boldsymbol{H}\boldsymbol{b}_1) \tag{3a}$$

whereas for $n > 1$,

$$S_{p_n} = (\boldsymbol{u}_{n-1} - \boldsymbol{b}_n)^T \boldsymbol{U}_{n-1} (\boldsymbol{u}_{n-1} - \boldsymbol{b}_n)$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{p} U_{l,k,n-1} (u_{l,n-1} - b_{l,n})(u_{k,n-1} - b_{k,n}) \tag{3b}$$

The prior information/regularization term has two functions. Regularization, Eq. (3a), permits a solution, which may not be accurate, when little information or few measurements are available. Regularization is needed to start and allow the sequential process to proceed when it is not possible to estimate all parameters from a single experiment. Hence regularization controls variations in the parameters, which otherwise could not be estimated, until enough information is available; i.e., a sufficient number of experiments have been considered to estimate all parameters. It also allows for prior information, Eq. (3b), about the parameters from a previous analysis to be included in the present analysis. The prior information is weighted accordingly with $\boldsymbol{U}_{n-1}$. Weighting is based on the previous analysis and discussed below.

To estimate thermal properties the difference between the measured and calculated temperatures in the augmented sum-of-squares function in Eq. (2) is minimized with respect to the thermal properties. The procedure takes the derivative with respect to the thermal properties, sets it equal to zero, and solves the resulting set of equations for the estimated properties $\hat{\boldsymbol{b}}_n$. The details of this procedure for a sequential analysis are discussed in Beck and Osman[5] and are outlined here for completeness and to physically motivate its use. The sequential procedure analyzes the first experiment ($n = 1$) without prior information, but regularization is included, Eq. (3a). The resulting set of equations for estimating parameters from the first experiment are[5]

$$n = 1$$

$$\boldsymbol{U}_1^{(k)} \left[\hat{\boldsymbol{b}}_1^{(k+1)} - \hat{\boldsymbol{b}}_1^{(k)}\right] = \boldsymbol{X}_1^{(k)T} \boldsymbol{W}_1 \left[\boldsymbol{Y}_1 - \hat{\boldsymbol{T}}_1^{(k)}\right] - \boldsymbol{H}^T \boldsymbol{H} \hat{\boldsymbol{b}}_1^{(k)} \tag{4a}$$

$$\boldsymbol{U}_1^{(k)} \equiv \boldsymbol{X}_1^{(k)T} \boldsymbol{W}_1 \boldsymbol{X}_1^{(k)} + \boldsymbol{H}^T \boldsymbol{H} \tag{4b}$$

The regularization term, $\boldsymbol{H}^T\boldsymbol{H}$ in Eq. (4), is discussed in Sec. IV, including how to select its magnitude.

Subsequent experiments are analyzed including information from the previous experiments:

$$n > 1$$

$$\boldsymbol{U}_n^{(k)} \left[\hat{\boldsymbol{b}}_n^{(k+1)} - \hat{\boldsymbol{b}}_n^{(k)}\right] = \boldsymbol{X}_n^{(k)T} \boldsymbol{W}_n \left[\boldsymbol{Y}_n - \hat{\boldsymbol{T}}_n^{(k)}\right] + \boldsymbol{U}_{n-1} \left[\boldsymbol{u}_{n-1} - \hat{\boldsymbol{b}}_n^{(k)}\right] \tag{5a}$$

$$\boldsymbol{U}_n^{(k)} \equiv \boldsymbol{X}_n^{(k)T} \boldsymbol{W}_n \boldsymbol{X}_n^{(k)} + \boldsymbol{U}_{n-1} \tag{5b}$$

The prior information and its weighting in Eq. (5a) are based on the converged values of the thermal properties from the previous analysis:

$$\boldsymbol{u}_{n-1} = \hat{\boldsymbol{b}}_{n-1} \tag{6}$$

$$\boldsymbol{U}_{n-1} = \boldsymbol{U}_{n-1}^{(k)} \Big|_{\boldsymbol{b} = \hat{\boldsymbol{b}}_{n-1}^{(k)}} \tag{7}$$

The subscript $n-1$ for the prior information and its weighting indicate that it is calculated from the analysis of experiment $n-1$, which includes results from all previous experiments.

The sensitivity matrix in Eqs. (4) and (5) is the partial derivative of the temperature with respect to the parameter vector

$$\boldsymbol{X}_n^{(k)} = \frac{\partial \boldsymbol{T}_n}{\partial \boldsymbol{b}} \Bigg|_{\boldsymbol{b} = \hat{\boldsymbol{b}}_n^{(k)}} \tag{8}$$

In Eqs. (4) and (5) the superscript (k) is an iteration index. Because the sensitivity matrix in Eq. (8) depends on the parameters being estimated, it is a nonlinear estimation problem, and iteration is required.

Equations similar to those in Eq. (5) for the sequential analysis are obtained when two experiments are analyzed simultaneously with certain assumptions.[5] Hence, similar properties are obtained whether experiments are analyzed simultaneously or sequentially. The parameters are not adversely biased by performing a sequential analysis.

An important issue is selecting the magnitude and order of the regularization. The regularization term used is a form of Tikhonov regularization.[11] There are several possible orders of Tikhonov regularization. Because issues associated with selecting regularization are often problem dependent, discussion on this topic is done in the results section. However, an understanding of the general influence of the regularization and prior information is important for understanding the sequential analysis concept.

The role of the regularization term is to stabilize the solution. Stability is required to estimate parameters when possibly little or no information is available. The information matrix $\boldsymbol{X}^T\boldsymbol{W}\boldsymbol{X}$ gives an indication of the information available for a parameter. When there is not much information available for a parameter, the corresponding diagonal entry in this matrix is very small. In the analysis of the first experiment Eq. (4b), the regularization term ($\boldsymbol{H}^T\boldsymbol{H}$) is added to the information matrix $\boldsymbol{X}_1^{(k)T}\boldsymbol{W}_1\boldsymbol{X}_1^{(k)}$ to prevent a zero entry for all subsequent analyses. The regularization term added during analysis of the first experiment carries its effect to *all* subsequent analyses in matrix $\boldsymbol{U}_{n-1}$, Eq. (5b). Unless an experiment provides enough information, such that ($\boldsymbol{X}_n^{(k)T}\boldsymbol{W}_n\boldsymbol{X}_n^{(k)} > \boldsymbol{U}_{n-1}$), the estimates from an experiment will remain near the prior estimates u_{n-1}. Consequently, it is important to correctly specify the magnitude of the regularization at the beginning. The danger is in making the regularization too large. Procedures to select the regularization are outlined in the Sec. IV.

III. Experimental Aspects

Experimental Apparatus

The sequential analysis of a series of experiments is applied to estimate the thermal properties of carbon-carbon composite. The laboratory experiment measures transient heat flux and temperature for estimating thermal properties. The carbon-carbon composite is orthotropic by nature. To characterize its properties, experiments with one- and two-dimensional heat flow are conducted. The one-dimensional experiments characterize the thermal conductivity in one principle direction. Two-dimensional experiments depend on two components of thermal conductivity. Because transient experiments are conducted, both experiments depend on the volumetric heat capacity. The series of experiments conducted cover the range from room temperature to 500°C. Previous analyses have considered the experiments independently and neglect variation in the thermal properties with temperature during an individual experiment[1,2]; a temperature rise of 20°C is typical for an individual experiment. In a sequential analysis the discrete experiments are joined during the analysis. Although the previous assumption[1,2] of constant thermal properties for each experiment are valid for the given temperature change, application of the sequential analysis demonstrates the approach. Consistency of the estimates from a sequential analysis,

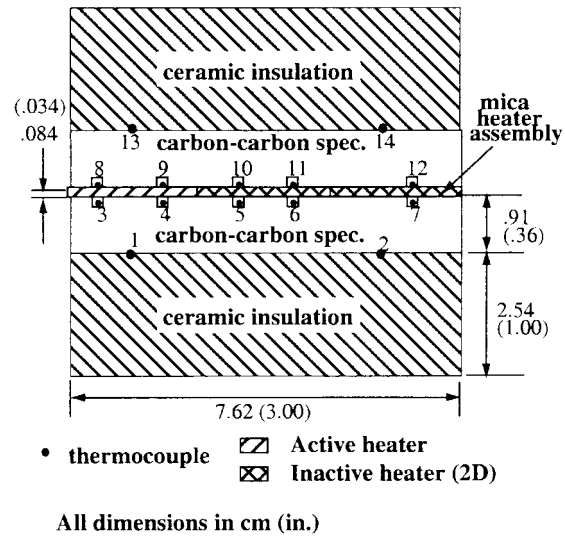

Fig. 1 Experimental apparatus for estimating thermal properties of carbon-carbon composite.

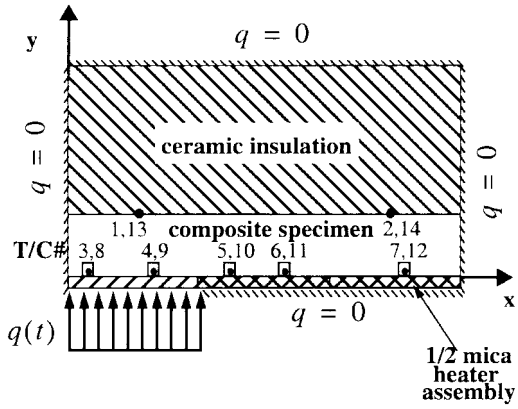

Fig. 2 Thermal model of two-dimensional experiment to estimate thermal properties of carbon-carbon. For one-dimensional experiment heat flux $q(t)$ extends over entire lower surface.

compared with an independent analysis of each discrete experiment, is supported in this study also.

A sketch of the experimental setup used to estimate the thermal properties of the carbon-carbon material is shown in Fig. 1. It consists of two nominally identical carbon-carbon composite specimens ($7.62 \times 7.62 \times 0.91$ cm) and ceramic insulation blocks ($7.62 \times 7.62 \times 2.54$ cm, Zircar Products Inc., Florida, New York) with a mica heater assembly [Thermal Circuits, Inc., Salem, Massachusetts; $\Omega(T_{\text{room}}) = 33$ ohms] located between the halves. The heater assembly consists of three (2.34×7.62 cm surface area) independently controlled heaters that extend over the entire surface normal to the page for negligible temperature variation in this direction. For two-dimensional experiments only one of the three heaters is activated. Five thermocouples (Type E, 0.254-mm nominal wire diameter) are embedded on the surface of each carbon-carbon specimen at the heater/specimen interface. The thermocouples (insulation removed) are cemented into grooves (0.381 mm wide $\times$0.457 mm deep) that extend the length of the specimen (normal to the page). Two thermocouples are at each interface of the carbon-carbon specimen and the ceramic insulation. The entire setup is mounted between two 3.18-mm-thick aluminum plates that are connected with threaded rods and hold the layers firmly in place. The entire apparatus is put in a furnace, which allows the initial temperature to be varied. Experimental aspects are further discussed in Ulbrich et al.[12]

A schematic of the thermal model of the experiment is shown in Fig. 2. All outer surfaces are assumed adiabatic, except for the surface where the energy is introduced by the heater. The energy to the heater is assumed to divide equally between the two halves and emanate from the middle of the heater assembly ($y = -0.042$ cm). Sensor locations are listed in Table 1. Sensor readings on opposite sides of the heater are averaged for the analysis. The model includes a mica heater and ceramic insulation in addition to the carbon-carbon. Separate experiments were conducted to estimate effective properties for the mica and insulation materials.[12]

Table 1 Sensor locations in experimental apparatus (origin at lower-left-hand corner of carbon-carbon specimen)

Sensor number	Location cm (in.)	
	x	y
3, 8	0.89 (0.35)	0
4, 9	1.91 (0.75)	0
5, 10	3.18 (1.25)	0
6, 11	4.45 (1.75)	0
7, 12	6.73 (2.65)	0
1, 13	1.27 (0.5)	0.91 (0.36)
2, 14	6.35 (2.5)	0.91 (0.36)

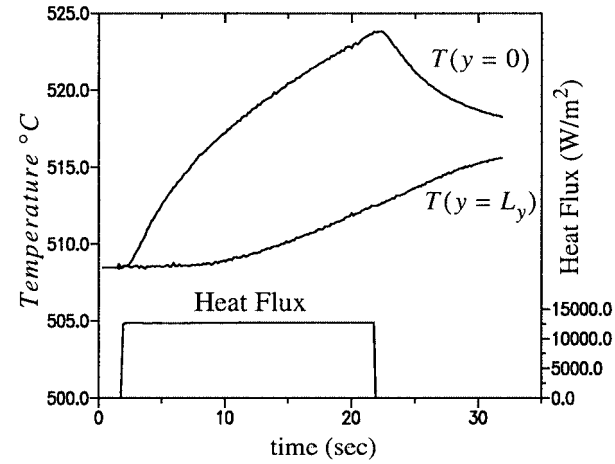

Fig. 3 Experimental data with one-dimensional heat flow (T_i = 508°C).

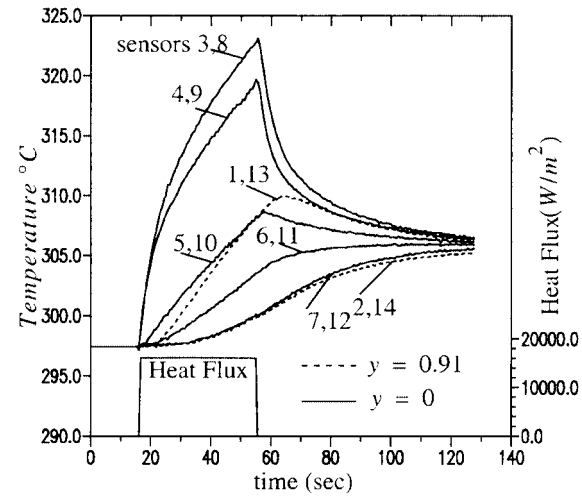

Fig. 4 Experimental data with two-dimensional heat flow (T_i = 297°C).

Experimental Data

Representative data from experiments with one-dimensional and two-dimensional heat flow are shown in Figs. 3 and 4, respectively. One-dimensional experiments average all thermocouple readings

along a constant y location. Two-dimensional experiments average thermocouples at symmetric locations. The one- and two-dimensional experiments have uniform initial temperatures of 508 and 297°C, respectively. Experimental data at other initial temperatures have the same general shape and features, but magnitudes are different depending on the initial temperature.

One-dimensional experiments were conducted at initial temperatures of 31, 42, 95, 109, 143, 159, 195, 259, 295, 304, 403, 455, and 508°C. Two-dimensional experiments were conducted at 65, 111, 158, 205, 256, 297, 353, and 403°C. There are 13 one-dimensional and 8 two-dimensional experiments.

IV. Results and Discussion

Description of Thermal Properties

Temperature dependence of the thermal properties is represented with a series of linear segments

$$k(T) = \sum_{j=1}^{p} k_j \theta_j(T) \tag{9}$$

$$\rho C(T) = \sum_{j=1}^{p} \rho C_j \theta_j(T) \tag{10}$$

where

$$\theta_j(T) \equiv 1 - \theta_{j+1} \quad T_j \le T \le T_{j+1} \tag{11a}$$

$$\theta_{j+1}(T) \equiv \frac{T - T_j}{T_{j+1} - T_j} \quad T_j \le T \le T_{j+1} \tag{11b}$$

$$\theta_j(T) = \theta_{j+1}(T) = 0 \quad \text{otherwise} \tag{11c}$$

This representation is selected for the thermal properties because most computational codes allow this input format. In addition, the parameters are the values of thermal properties at the reference temperature:

$$k_j \equiv k(T_j) \tag{12a}$$

$$\rho C_j \equiv \rho C(T_j) \tag{12b}$$

Hence the parameter values have physical meaning. In contrast, other investigations[6,9] have estimated an intercept and slope for the linear segment. The slope of a property is not a value for which engineers typically have a physical appreciation. Also, sensitivity to the slope and intercept can have quite different magnitudes, making them difficult to simultaneously estimate.

It is not required that values of conductivity and volumetric heat capacity are estimated at the same referenced temperatures T_j. Conductivity could be estimated at different reference temperatures than the volumetric heat capacity. However, the same reference temperatures are used to describe the temperature dependence for this material. Experiments are available at intervals spanning the range from room temperature to approximately 500°C. Past analysis of these experiments[1,2] has indicated the temperature dependence of the properties is second order. In this investigation two linear segments ($p = 3$) are estimated to represent the temperature dependence of the properties. The general representation is shown in Fig. 5. Three parameters represent two linear segments to describe the temperature dependence. In general it is best to estimate as few parameters as needed to describe the property variation. For one-dimensional experiments six parameters are estimated to describe k_y and ρC, whereas nine parameters are required for k_y, k_x, and ρC in two-dimensional experiments.

Sequential Analysis

The idea in a sequential analysis is to control the variation of the parameters that are difficult to estimate with regularization. For example, it is obviously not possible to accurately estimate the thermal conductivity at 250°C, when the experiment reaches a maximum temperature of 50°C. The difficulty of estimating a component is gauged by the sensitivity matrix ($\boldsymbol{X}$). If the entry along the diagonal of the information matrix ($\boldsymbol{X}^T \boldsymbol{W} \boldsymbol{X}$) for a parameter is small compared to other entries along the diagonal (for the same property), it may be difficult to estimate that parameter. It is for such parameters that the regularization term stabilizes the solution and permits a solution even though the particular experiment is insensitive to possibly several parameters. Subsequent experiments in the sequential analysis provide information about the parameters for which the current experiment may be insensitive.

Table 2 Scaled sensitivity coefficients for one-dimensional experiments at three initial temperatures, $t = 20$ s and $y = 0$

Coefficient	$T_{k_{y,1}}$, 30°C	$T_{k_{y,2}}$, 260°C	$T_{k_{y,3}}$, 525°C
Exp 1, $T_i = 30$°C	−5.5	−0.3	0.0
Exp 2, $T_i = 259$°C	−0.7E−04	−4.1	−0.6E−01
Exp 3, $T_i = 508$°C	0.0	−0.2	−6.2

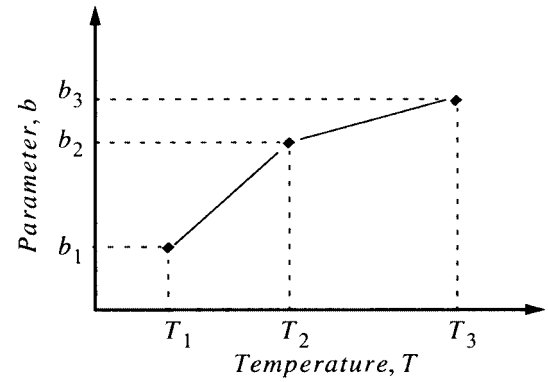

Fig. 5 Temperature dependence for a property described by two linear segments (three components).

Sensitivity coefficients can provide insight to the sequential procedure. Scaled sensitivity coefficients are particularly useful as they have units of temperature and can be readily compared for different parameters. The scaled sensitivity coefficient for parameter b_i is defined as

$$T_{b_i} = b_i \frac{\partial T}{\partial b_i} \tag{13}$$

There are several methods available to compute sensitivity coefficients.[13] For an assumed $k_y(T)$ relationship that is composed of two linear segments, there are three values of conductivity to be estimated, corresponding to the reference temperatures $T_1 = 30$°C, $T_2 = 260$°C, and $T_3 = 525$°C, with $k_{y,1} = k_y(T = T_1)$, $k_{y,2} = k_y(T = T_2)$, and $k_{y,3} = k_y(T = T_3)$. The choice of these reference temperatures is arbitrary but is influenced by the nominal temperature of each of the three experiments used in the analysis. The fact that there are three experiments and three reference temperatures is coincidental. This is a subset of the experiments selected to illustrate the concept. The three experiments will each have three scaled sensitivity coefficients for a total of nine sensitivity coefficients. These nine sensitivity coefficients are arranged in matrix form in Table 2 for the sensor location nearest the heater ($y = 0$) and $t = 20$ s.

The largest sensitivity coefficients are located along the diagonal of Table 2. Physically, this is because each experiment is designed to operate around a temperature associated with each thermal conductivity component. Experiment 1 gives no information about $k_{y,3}$; similarly, Experiment 3 gives no information about $k_{y,1}$. These nine sensitivity coefficients are shown graphically as a function of time in Fig. 6. (Components $k_{y,3}$ and $k_{y,1}$ are identically zero and not shown for Experiments 1 and 3, respectively, in Fig. 6.) From the sensitivity we see that it is impossible to accurately estimate all three components from a single experiment. However, the sensitivity coefficients

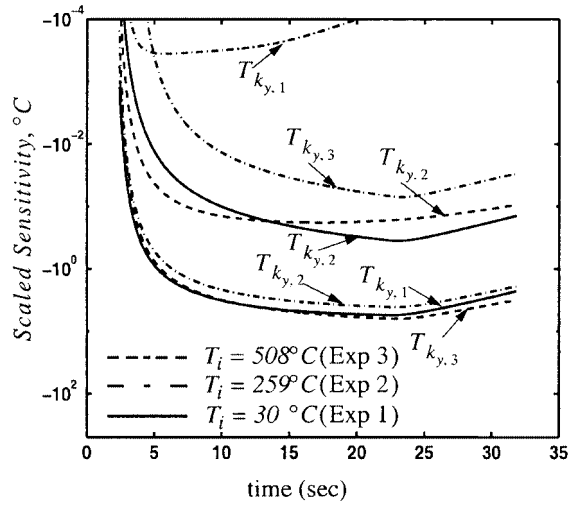

Fig. 6 Scaled sensitivity coefficients for three components of thermal conductivity (k_y) representing its temperature dependence; three one-dimensional experiments with different initial temperatures.

are large for the component near the range of the experiment (see Table 2). Other sensitivities shown are one or more orders of magnitude smaller than the components with a large sensitivity. Consequently, by combining experiments the temperature dependence can be accurately estimated. Similar trends could be shown for the sensitivity to other thermal properties.

In Sec. II a sequential analysis proceeds to estimate components prior to sufficient information being available. This is accomplished using regularization. The regularization term is included in the analysis of the first experiment, but it affects all subsequent analyses. Its effect is carried forward in the weighting of the prior information. The regularization term, $\boldsymbol{H}^T\boldsymbol{H}$ in Eq. (4), is a form of Tikhonov regularization. Several orders of Tikhonov regularization are possible.[14] As shown in Fig. 5, it is intended to represent the temperature dependence of the thermal properties with a series of linear segments. Because this is the case, first-order regularization is used. First order is chosen because it takes the difference between the two components describing the end points of the linear segment. The nature of first-order regularization is to minimize the difference between these end points, i.e., minimize the slope. In the limit as the regularization magnitude becomes large, the end points describing the line become equal; the linear segment becomes a constant.

For first-order Tikhonov regularization when estimating two linear segments (three components) to represent the conductivity and volumetric heat capacity, the regularization matrix ($\boldsymbol{H}$) contains the weighting for differences between successive parameters. For example, it would have entries to weight the difference between b_1 and b_2 and b_2 and b_3, which describe the piecewise linear temperature dependence in Fig. 5. The regularization matrix for estimating parameters to describe two properties, both with temperature dependence represented as shown in Fig. 5, is

$$\boldsymbol{H} = \begin{bmatrix} -\sqrt{\alpha_1} & \sqrt{\alpha_1} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\alpha_2} & \sqrt{\alpha_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\alpha_4} & \sqrt{\alpha_4} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\alpha_5} & \sqrt{\alpha_5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

The weights in the regularization matrix correspond to the parameter vector. The first three parameters (b_1 to b_3) represent two linear functions to describe the temperature dependence of the first property. The next three are the parameters to describe temperature dependence of the second property. The weighting for the third row is zero; regularization between parameters describing two different thermal properties is meaningless.

To clarify, consider the following parameter vector

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = \begin{bmatrix} \rho C_1 = \rho C(T = T_1) \\ \rho C_2 = \rho C(T = T_2) \\ \rho C_3 = \rho C(T = T_3) \\ k_{y,1} = k_y(T = T_1) \\ k_{y,2} = k_y(T = T_2) \\ k_{y,3} = k_y(T = T_3) \end{bmatrix} \tag{15}$$

where T_1, T_2, and T_3 are successive increasing temperatures. In Eq. (14) α_1 is the weighting of the regularization term that includes the difference between the end points (ρC_1 and ρC_2) of the first linear segment for ρC. Likewise, α_2 is the weighting of the regularization of the second linear segment (ρC_2 and ρC_3), and α_4 and α_5 are the regularization weights for the linear segments representing k_y. These weights must be selected appropriately so that they control variations in the estimated parameters during the analysis of initial experiments, but do not influence the final estimated properties (after all experiments have been considered). To select the magnitudes of the regularization weights, the sensitivity matrix is examined. For the regularization term not to influence the estimated thermal properties, the magnitudes of α_1, α_2, α_4, and α_5 are selected such that the diagonal of $\boldsymbol{H}^T\boldsymbol{H}$ in Eq. (4b) is small when compared to diagonal of $\boldsymbol{X}_n^T\boldsymbol{W}_n\boldsymbol{X}_n$ in Eq. (5b) after all experiments have been analyzed. Provided the regularization $\boldsymbol{H}^T\boldsymbol{H}$ is 3–4 orders of magnitude smaller than $\boldsymbol{X}_n^T\boldsymbol{W}_n\boldsymbol{X}_n$, the estimated properties are insensitive to the specified value for the regularization. A very rough estimate of the magnitude for the jth component of the main diagonal can be obtained as $\boldsymbol{X}_n^T\boldsymbol{X}_n \approx (\Delta T_n / b_j)^2$, where ΔT_n is the temperature change for experiment n and b_j is the jth parameter. This can help select an initial magnitude for $\boldsymbol{H}^T\boldsymbol{H}$.

Numerical Implementation

The procedure to perform the sequential analysis described in the Analysis (Sec. II) are implemented in the parameter estimation code PROP2D. The code combines a two-dimensional finite element-based solver[15] with sequential parameter estimation techniques. Sensitivity coefficients are computed using a finite difference[13] approximation.

The discretization of experimental apparatus, including a grid study, is discussed in Dowding and Blackwell.[16] One-dimensional solutions are obtained with the two-dimensional code.

Estimated Thermal Properties

One- and two-dimensional experiments are available to estimate the thermal properties of the carbon-carbon. The one- and two-dimensional experiments are considered separately to demonstrate consistency between the experiments and check the adequacy of the orthotropic material model. Then the one- and two-dimensional experiments are combined using the sequential procedure. Estimated properties from the sequential analysis are contrasted with previous analyses that considered the experiments independently, assuming the properties are constant for each experiment.

There are 13 one-dimensional and 8 two-dimensional experiments. The open symbols shown in Fig. 7 represent the estimated properties (k_y, ρC—one-dimensional and k_y, k_x, ρC—two-dimensional) when analyzing each experiment independently,[1,2] assuming thermal properties are constant for an experiment, but vary between experiments. The circles are the independent analysis of one-dimensional experiments and squares are for two-dimensional experiments.

Also shown in Fig. 7 are estimated linearly varying segments from the sequential analysis. The dash-dot line is the result of a sequential analysis combining all 13 one-dimensional experiments to estimate two linearly varying segments that represent the temperature dependence of the thermal properties. The dash-dash line represents the sequential analysis of the eight two-dimensional experiments. Notice that the prescribed end point temperatures of the linear segments are slightly different for the one- and two-dimensional experiments;

Table 3 Estimated end points for two linearly varying segments from a sequential analysis to describe temperature-dependent thermal properties

Temp, °C	ρC $(b_1 - b_3)$, J/m^3°C	k_y $(b_4 - b_6)$, W/m°C	k_x $(b_7 - b_9)$, W/m°C
	One-dimensional experiments		
	$(\alpha_1 = \alpha_2 = 0.5E-12)$, $(\alpha_4 = \alpha_5 = 0.25)$		
30	1.41 ± 0.003	3.35 ± 0.013	——
260	2.56 ± 0.006	4.60 ± 0.019	——
525	3.06 ± 0.009	4.85 ± 0.022	——
	Two-dimensional experiments		
	$(\alpha_1 = \alpha_2 = 0.5E-12)$, $(\alpha_4 = \alpha_5 = 1.0)$, $(\alpha_7 = \alpha_8 = 0.05)$		
65	1.52 ± 0.008	3.82 ± 0.078	59.5 ± 0.89
250	2.29 ± 0.008	4.76 ± 0.056	61.2 ± 0.59
430	2.71 ± 0.012	5.17 ± 0.065	56.9 ± 0.68
	One- and two-dimensional experiments		
	$(\alpha_1 = \alpha_2 = 0.5E-12)$, $(\alpha_4 = \alpha_5 = 0.25)$, $(\alpha_7 = \alpha_8 = 0.05)$		
30	1.41 ± 0.004	3.32 ± 0.021	61.0 ± 1.0
260	2.34 ± 0.007	4.81 ± 0.034	60.5 ± 0.57
525	2.98 ± 0.016	5.01 ± 0.052	55.7 ± 1.22

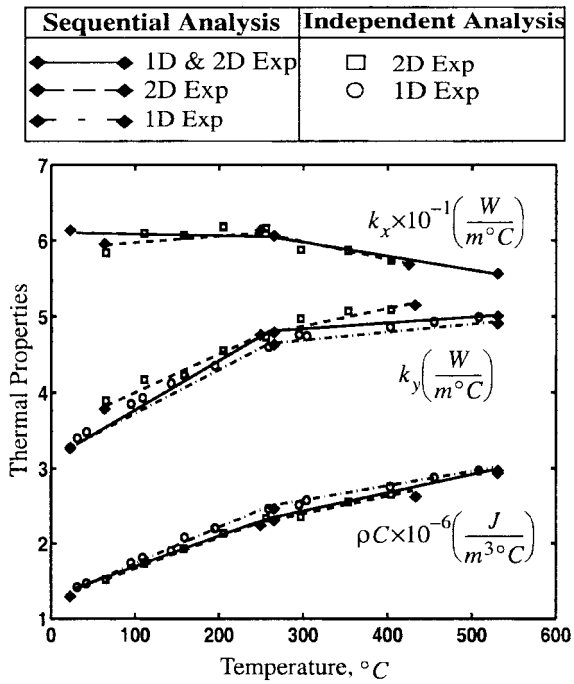

Fig. 7 Estimated thermal properties considering each experiment independently, assuming thermal properties are constant for each experiment and for a sequential analysis joining the experiments during the analysis to estimate temperature-dependent thermal properties.

one-dimensional experiments extend over a larger temperature range than two-dimensional experiments. The estimated end points of the linear segments from one- and two-dimensional experiments, with confidence intervals (discussed next), are given in the top two blocks of Table 3.

A unique possibility of a sequential analysis is the joining of unrelated experiments. This aspect is used to combine the one- and two-dimensional experiments to estimate a single temperature function for each property. To accomplish this, the estimated parameters from sequentially analyzing the one-dimensional experiments are used as the prior information to begin the sequential analysis of the two-dimensional experiments. Because one-dimensional experiments estimate k_y and ρC (for a total of six parameters), before beginning the two-dimensional sequential analysis (estimating nine parameters) the prior information weighting matrix needs to be augmented. The additional information added to the prior information weighting matrix are the terms for regularizing k_x. To add regularization at this time for k_y and ρC is not required because it has already been added at the start of the sequential analysis of the one-dimensional experiments. The results of a sequential analysis that combines the one- and two-dimensional experiments to estimate a single temperature-dependent function for each property are shown as the solid line in Fig. 7. The estimated linear functions using one- and two-dimensional experiments are given in the bottom block of Table 3.

Figure 7 and Table 3 contain a great deal of information. There are results from the 21 independent and three sequential analyses. The linear functions estimated from the one- and two-dimensional experiments separately show good agreement. The linear segments estimated with a sequential analysis of two-dimensional experiments (dash-dash line) are above the segments from the sequential analysis of one-dimensional experiments (dash-dot line) for k_y; the opposite relationship is shown for ρC. There is better agreement of the one- and two-dimensional sequential analysis for ρC than for k_y. The largest deviation is near high temperatures for k_y. The slopes for the two lines are most different there. The difference is because of the limited two-dimensional data available at the higher temperatures. A comparison of estimated properties from the independent analysis of one- and two-dimensional experiments (open symbol in Fig. 7) show a maximum deviation of 6–7%. The differences between properties estimated with one- and two-dimensional experiments are within the total experimental uncertainty.[17]

Because estimated values are available from one- and two-dimensional experiments separately (for k_y and ρC), this begs the question of which values, or combination of estimated values, to use to describe properties. All experimental data points are not necessarily equal. Some experiments have larger sensitivity coefficients for certain parameters; hence, these experiments should be weighted more than those with smaller sensitivity. A sequential analysis can combine the analysis of the one- and two-dimensional experiments. In the sequential procedure experiments with larger sensitivity coefficients are automatically weighted proportionally more than those experiments with smaller sensitivity through the prior information weighting.

This point is reflected in the sequential analysis that combines one- and two-dimensional experiments (solid line in Fig. 7). For example, at the lower temperatures for k_y the estimated line from the sequential analysis of combined one- and two-dimensional experiments does not split the difference between the sequentially estimated lines using one- and two-dimensional experiments individually. Because there is a cluster of one-dimensional experiments near the prescribed end point of this linear segment, it lies closer to the end point from the sequential analysis of one-dimensional experiments. These experiments have a larger sensitivity (for the lower end point) than higher temperature experiments and are most influential on the end point of the linear segment for a sequential analysis. Consequently the sequential analysis of one-dimensional, and combined one- and two-dimensional experiments, show close agreement at lower temperatures for k_y. The line from the combined one- and two-dimensional experiments is closer to the two-dimensional results for ρC. This is believed to be the result of the two-dimensional experiments having larger sensitivity to ρC than the one-dimensional.[17] A curve fit of the properties would weight all experiments equally, unless weighted least squares is used.

The properties estimated from an independent analysis are predominantly above the linear functions estimated with a sequential analysis in Fig. 7: this is to be expected. The properties from the independent analysis are plotted at the initial temperature of the experiment. A better representative temperature is some indicative average of the experiment and higher than the initial temperature. The expectation is that the property values for the independent analysis should be shifted to a higher temperature to compare with the linear segments. The property change with temperature is greatest for k_y, and it shows this effect most. This shift would produce a closer agreement between the independent and sequential results in Fig. 7.

Included in Table 3 are confidence intervals (95%) for the end points of the linear segments describing the temperature dependence of the thermal properties. The confidence intervals are calculated assuming errors in temperature measurements only. The errors are

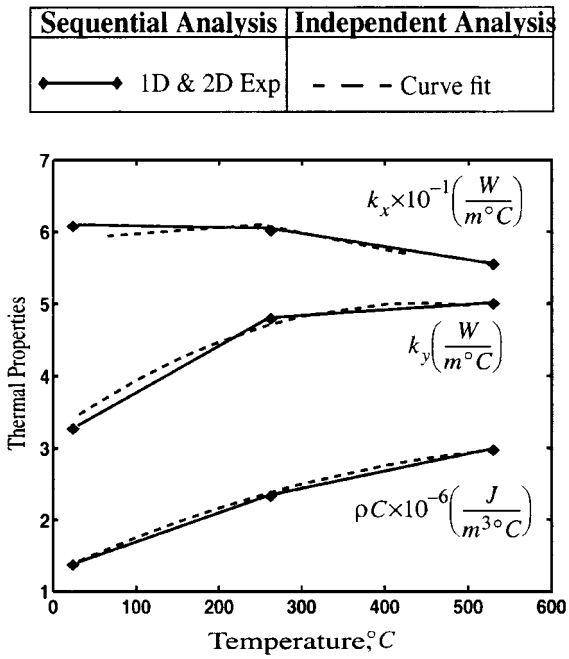

Fig. 8 Comparison of sequentially estimated temperature-dependent thermal properties with a curve fit of the independent analyses (one- and two-dimensional experiments).

assumed normal, uncorrelated, with zero mean. These assumptions produce conservative approximations; nevertheless, there is insight available.

Confidence intervals for the linear end points from the analysis of two-dimensional experiments are larger than those for one-dimensional experiments. The confidence intervals may be as much as six times as large for two-dimensional experiments, but the magnitude is only 2% of the property value. When both one- and two-dimensional experiments are considered, the confidence intervals increase with temperature (relative to the property value), with the exception of k_x. For k_x the confidence interval for the value at the prescribed midtemperature is the smallest because of the limited information available at the minimum and maximum temperatures for k_x.

To describe the temperature dependence when the experiments are analyzed independently, the results (one- and two-dimensional combined) are curve fit.[1] To perform a curve fit, a reference temperature is associated with the estimated properties for each experiment. In this case the initial temperature is used as the reference temperature. To curve fit, the k_x independent results were assumed to vary linearly, while an F-test[18] indicated a second-order dependence is needed to represent k_y and ρC. A comparison of the curve fit to the estimated independent properties with the sequentially estimated linear segments is shown in Fig. 8. The sequentially estimated linear functions closely approximate the curve fits of the properties estimated from an independent analysis. As was already noted, the values for an independent analysis lie above those for a sequential analysis. The curve fit functions are shifted along the temperature axis. The agreement demonstrates that a sequential analysis results in comparable estimated properties.

Selecting the correct magnitude of the regularization and the prescribed end point temperature values to represent the temperature dependence of the properties are important issues in applying the sequential analysis. Procedures to select the magnitude of the regularization parameters were discussed in the preceding section. The magnitudes listed in Table 3 were reduced several orders of magnitude to demonstrate that the values are not influential on the final estimated values. For convenience the prescribed end points of the linear temperature dependence were selected to be at equal intervals over the temperature range. The midrange temperature is slightly adjusted to be near a value for which there is an experiment with the same nominal temperature.

Noted by a reviewer was the extension of the current methods to three dimensions. The current material had conductivity that differed in directions normal and parallel to the fiber direction. Fibers are normal to the y direction in Fig. 2. The manufacturer of the carbon-carbon suggested conductivity parallel to the fibers (along the x and z direction) was similar. Though not rigorously tested, experiments demonstrated this to be the case.

If the material were fully orthotropic (three components of conductivity), one could imagine an experiment that heats over only a portion of surface to produce three-dimensional heat flow. A three-dimensional experiment would require carefully studying sensitivity coefficients prior to the experiment to investigate sensor placement, heating area, heating duration, experiment duration, etc. A sequential analysis would allow experiments of increasing complexity to be combined. A suite of experiments could progress from one-dimensional experiments, to two separate two-dimensional experiments investigating two components of conductivity, culminating in a full three-dimensional experiment to study all three components of conductivity. Whereas initially the experiments could be studied independently to perfect the techniques, in the end a sequential analysis could combine the experiments.

V. Conclusions

A sequential procedure was presented that allows experiments, possibly different in nature, to be combined during the analysis. The sequential approach was applied to estimate temperature-dependent thermal properties of carbon-carbon composite from a series of experiments covering a temperature range from room temperature to 500°C. The temperature dependence estimated with a sequential analysis was compared with an analysis that considered the experiment independently, assuming constant properties that varied between experiments.

The sequential analysis produced temperature dependence within 2–3% of the independent analysis (plotted at the initial temperature) for the thermal properties. Experiments can be analyzed in a sequential manner and without affecting the estimated thermal properties. Using a sequential analysis, possibly fewer experiments can be conducted and accurately capture the temperature dependence.

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References

[1]Dowding, K. J., Beck, J. V., and Blackwell, B. F., "Estimation of Directional-Dependent Thermal Properties in a Carbon-Carbon Composite," *International Journal of Heat and Mass Transfer*, Vol. 39, No. 15, 1996, pp. 3157–3164.

[2]Dowding, K., Beck, J., Ulbrich, A., Blackwell, B., and Hayes, J., "Estimation of Thermal Properties and Surface Heat Flux in a Carbon-Carbon Composite Material," *Journal of Thermophysics and Heat Transfer*, Vol. 9, No. 2, 1995, pp. 345–351.

[3]Loh, M., and Beck, J. V., "Simultaneous Estimation of Two Thermal Conductivity Components from Transient Two-Dimensional Experiments," American Society of Mechanical Engineers Winter Annual Meeting, Paper 91-WA/HT-11, Atlanta, GA, Dec. 1991.

[4]Woodbury, K. A., and Boohaker, C. G., "Simultaneous Determination of Temperature-Dependent Thermal Conductivity and Volumetric Heat Capacity by an Inverse Technique," *Proceedings of the American Society of Mechanical Engineers Heat Transfer Division*, edited by A. Gopinath, S. S. Sadhal, P. D. Jones, J. Seyed-Yagoobi, and K. A. Woodbury, HTD-Vol. 332, Vol. 1, American Society of Mechanical Engineers, New York, 1996, pp. 269–274.

[5]Beck, J. V., and Osman, A. M., "Sequential Estimation of Temperature-Dependent Thermal Properties," *High Temperature-High Pressure*, Vol. 23, No. 3, 1991, pp. 255–266.

[6]Huang, C. H., and Ozisik, M. N., "Direct Integration Approach for Simultaneously Estimating Temperature-Dependent Thermal Conductivity and Heat Capacity," *Numerical Heat Transfer*, Pt. A, Vol. 20, No. 1, 1991, pp. 95–110.

[7]Huang, C. H., and Yuan, Y. Y., "An Inverse Problem in Simultaneously Measuring Temperature-Dependent Thermal Conductivity and Heat Capacity," *International Journal of Heat and Mass Transfer*, Vol. 38, No. 18, 1995, pp. 3433–3441.

[8]Flach, G. P., and Ozisik, M. N., "Inverse Heat Conduction Problem of Simultaneously Estimating Spatially Varying Thermal Conductivity and Heat Capacity per Unit Volume," *Numerical Heat Transfer*, Pt. A, Vol. 16, No. 2, 1989, pp. 249–266.

[9]Sawaf, B., Ozisik, M. N., and Jarny, Y., "An Inverse Analysis to Estimate Linearly Temperature-Dependent Thermal Conductivity Components and Heat Capacity of an Orthotropic Medium," *International Journal of Heat and Mass Transfer*, Vol. 38, No. 16, 1995, pp. 3005–3010.

[10]Eldred, M. S., Outka, D. E., Bohnhoff, W. J., Witkowski, W. R., Romero, V. J., Ponslet, E. R., and Chen, K. S., "Optimization of Complex Mechanics Simulations with Object-Oriented Software Design," *Computer Modeling and Simulation in Engineering*, Vol. 1, No. 3, 1996, pp. 323–352.

[11]Tikhonov, A. N., and Arsenin, V. Y., *Solutions of Ill-Posed Problems*, V. H. Winston and Sons, Washington, DC, 1977, pp. 1–50.

[12]Ulbrich, A., Beck, J. V., and Dowding, K., "Measurements of Transient Temperatures and Heat Fluxes in a Composite Material for the Estimation of Thermal Properties—Part I: Experimental Aspects," Michigan State Univ., TRMSU-ENGR-009-93, East Lansing, MI, Dec. 1993.

[13]Blackwell, B. F., Cochran, R. J., and Dowding, K. J., "Development and Implementation of Sensitivity Coefficient Equations for Heat Conduction Problems," *American Society of Mechanical Engineers Proceedings of the AIAA/ASME 7th Joint Thermophysics and Heat Transfer Conference*, Vol. 2, edited by R. S. Amano, B. F. Arnaly, T. S. Chen, A. Emery, J. A. Liburdy, D. P. Shatto, N. K. Anand, B. Blackwell, T. Y. Chu, J. Lage, O. Oosthuizen, and K. A. Woodbury, American Society of Mechanical Engineers, New York, 1998, pp. 303–316.

[14]Beck, J. V., Blackwell, B., and St. Clair, C. R., *Inverse Heat Conduction: Ill-Posed Problems*, Wiley, New York, 1985, pp. 138–140.

[15]Shapiro, A. B., "TOPAZ2D: A Two-Dimensional Finite Element Code for Heat Transfer Analysis, Electrostatic, and Magnetostatic Problems," Lawrence Livermore National Lab., Rept. UCID-20824, Livermore, CA, July 1986.

[16]Dowding, K. J., and Blackwell, B. F., "Joint Experimental/Computational Techniques to Measure Thermal Properties of Solids," *Measurement Science and Technology*, Vol. 9, No. 6, 1998, pp. 877–887.

[17]Dowding, K. J., "Multi-Dimensional Estimation of Thermal Properties and Surface Heat Flux Using Experimental Data and a Sequential Gradient Method," Ph.D. Dissertation, Dept. of Mechanical Engineering, Michigan State Univ., East Lansing, MI, 1997, Chaps. 3 and 4, pp. 6–63.

[18]Beck, J. V., and Arnold, K., *Parameter Estimation in Engineering and Science*, Wiley, New York, 1977, pp. 242–245.